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Symbolic computation and new families of exact soliton-like solutions to the integrable Broer–Kaup (BK) equations in (2 + 1)-dimensional spaces

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Abstract

In this paper, new families of soliton-like solutions are obtained for $(2 + 1)$-dimensional integrable Broer–Kaup equations by using the symbolic computation method developed by Gao and Tian. Sample solutions obtained from these methods are presented. Solitary wave solutions are merely a special case in one family. The method can also be extended to other types of nonlinear evolution equations in mathematical physics.

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1. Introduction

As is well known, many completely integrable models were presented during the course of studying shallow water waves. For example, KdV-type equations, the WBK equation, the integrable long wave equation, the Boussinesq equation, etc. Finding exact solutions, in particular soliton solutions, of these nonlinear evolution equations (NEEs) plays an important role in soliton theory [1–3]. Up to now, there exist many powerful methods to search for exact solutions of NEEs, such as the Backlund transformation, Darboux transformation, Hirota method, bilinear method, tanh method, etc [1–17]. But finding more powerful methods is still a significant subject in solving NEEs in soliton theory and its applications.

With the rapid development of computerized symbolic computation, the application of symbolic computation to the physical and mathematical sciences appears to have a bright future. There have recently appeared more studies on the exact solutions of KdV-type equations, the $(3+1)$-dimensional Jimbo–Miwa equation, $(2+1)$-dimensional breaking soliton equations, the $(2+1)$-dimensional dispersive long wave equation and the $(2+1)$-dimensional KPP equation, such as [9–15] and references therein.

For the integrable Broer–Kaup (BK) equations in $(2 + 1)$-dimensional spaces [4, 5]

$$\begin{aligned} &H_{ty} + 2(HH_x)_y + 2G_{xx} - H_{xxy} = 0 \\ &G_t + 2(GH)_x + G_{xx} = 0 \end{aligned} \qquad (1)$$

Painlevé properties and infinite many truncated symmetries with arbitrary function in t of the system (1) have recently been studied [8] by using both the WTC approach [6] and the formal series symmetry method [7]. In this paper, we would like to use symbolic computation to obtain new families of exact solutions to the integrable BK equation in (2 + 1)-dimensional spaces, and to give examples of these solutions obtained.

The rest of the paper is organized as follows. In section 2, a transformation of system (1) is presented. In section 3, many types of explicit exact soliton-like solutions, including solitary wave solutions, are obtained for system (1) by using symbolic computation. Finally, some conclusions are given in section 4.

2. Leading to a transformation for system (1)

The main ideas of the following two basic methods are important:

(a) Hirota's dependent variables transformation [16,17] introduces, to begin with, a dependent variable $\phi(x,t)$ with a differentiator acting on its function $f(\phi(x,t)) = \ln(\phi(x,t))$.

(b) The Clarkson–Kruskal method [18] considers a general function $F(x,t,\Phi(x,t))$ and tries to established an ordinary differential equation (ODE) for $\phi(x,t)$ so as to impose conditions upon F and ϕ.

Recently, based on the above two ideas, Gao and Tian presented a direct method that has been successfully applied to find soliton-like solutions of (2 + 1)-dimensional breaking soliton equations, the (2 + 1)-dimensional dispersive long wave equation and (2 + 1)-dimensional generalized shallow water wave equations [11–13].

In order to find exact solutions to system (1), based on the idea of the method developed by Gao and Tian, we assume that system (1) possesses the following solutions:

$$\begin{aligned} H(x,y,t) &= \partial_x^m \partial_y^n h[w(x,y,t)] + A \\ G(x,y,t) &= \partial_x^i \partial_y^j g[w(x,y,t)] + B \end{aligned} \tag{2}$$

where the integers m, n, i and j, as well as the constants A and B are to be determined later. The leading-order analysis is performed as follows: for the first equation of system (1), the (possible) highest-power terms are $w_x^{2m+1} w_y^{2n+1}$, $w_x^{i+2} w_y^{j}$ and $w_x^{m+2} w_y^{n+1}$, which are, respectively, contributed by $(HH_x)_y$, G_{xx} and H_{xxy}. Then the balancing act requires that these three terms have the same power, i.e. $2m+1 = i+2 = m+2$ and $2n+1 = j = n+1$. Similarly, for the second equation of system (1), the (possible) highest-power terms are $w_x^{m+i+1} w_y^{n+j}$ and $w_x^{i+2} w_y^{j}$, which are, respectively, contributed by $(GH)_x$ and G_{xx}. Then the balancing act requires that these two terms have the same power, i.e. $m+i+1 = i+2$ and $n+j = j$. Therefore, we conclude the analysis with $m = i = j = 1$ and $n = 0$, so as to get from system (2)

$$\begin{aligned} H(x,y,t) &= \frac{\partial}{\partial x} h[w(x,y,t)] + A = h' w_x + A \\ G(x,y,t) &= \frac{\partial^2}{\partial x \partial y} g[w(x,y,t)] + A = g'' w_x w_y + g' w_{xy} + B \end{aligned} \tag{3}$$

in which the functions $H(x,y,t)$ and $G(x,y,t)$ are expressed by two functions $h(w)$ and $g(w)$ of one argument $w(x,y,t)$ only. Where the function $h(w)$, $g(w)$ and the constants A, B are to be determined later, the prime denotes differentiation w.r.t. w.

3. Exact soliton-like solutions of system (1)

With the aid of symbolic computation software such as Mathematica or Maple, substituting system (3) into system (1), and collecting all the homogeneous terms in partial derivatives of $w(x, y, t)$, we get

$$\begin{aligned}&[-h'''' + 2(h''^2 + h'h''') + 2g'''']w_x^3 w_y - 3h''' w_x^2 w_{xy} - 3h''' w_x w_y w_{xx}\\ &\quad +2(3h'h'' w_x^2 w_{xy} + 2h'h'' w_x w_y w_{xx} + Ah''' w_x^2 w_y) + 6g''' w_x^2 w_{xy}\\ &\quad +6g''' w_x w_y w_{xx} + h''' w_x w_y w_t - 3h'' w_{xx} w_{xy} - 3h'' w_x w_{xxy} - h'' w_y w_{xxx}\\ &\quad +2(h'^2 w_{xx} w_{xy} + h'^2 w_x w_{xxy} + 2Ah'' w_x w_{xy} + Ah'' w_y w_{xx}) + 6g'' w_{xy} w_{xx}\\ &\quad +6g'' w_x w_{xxy} + 2g'' w_y w_{xxx} + h'' w_{xy} w_t + h'' w_x w_{yt} + h'' w_y w_{xt}\\ &\quad -h' w_{xxxy} + A w_{xxy} + 2g' w_{xxxy} + h' w_{xyt} = 0 \end{aligned} \tag{4}$$

$$\begin{aligned}&[g'''' + 2(h'g''' + h''g'')]w_x^3 w_y + 3g''' w_x^2 w_{xy} + 3g''' w_x w_y w_{xx}\\ &\quad +2(2h'g'' w_x^2 w_{xy} + 2h'g'' w_x w_y w_{xx} + Ag''' w_x^2 w_y + h''g' w_x^2 w_{xy}) + g''' w_x w_y w_t\\ &\quad +3g'' w_{xx} w_{xy} + 3g'' w_x w_{xxy} + g'' w_y w_{xxx} + 2(h'g' w_{xx} w_{xy} + h'^2 w_x w_{xxy}\\ &\quad +2Ag'' w_x w_{xy} + Ag'' w_y w_{xx} + Bh'' w_x^2) + g'' w_{xy} w_t + g'' w_x w_{yt} + g'' w_y w_{xt}\\ &\quad +g' w_{xxxy} + 2(Ag' w_{xxy} + g' w_{xyt} + Bh' w_{xx}) = 0. \end{aligned} \tag{5}$$

From equations (4) and (5), we can find that the nonlinear terms and the highest-order partial derivative terms in system (1) have been balanced by using system (3). To determine the functions $h(w)$ and $g(w)$, we set the coefficients of the term $w_x^3 w_y$ in equations (4) and (5) to zero, respectively, which yield a system of ODEs with respect to $h(w)$ and $g(w)$:

$$\begin{aligned}&-h'''' + 2(h''^2 + h'h''') + 2g'''' = 0\\ &g'''' + 2(h'g''' + h''g'') = 0\end{aligned} \tag{6}$$

which have the special solutions in the form

$$h[w(x, y, t)] = g[w(x, y, t)] = \ln w(x, y, t) \tag{7}$$

which implies that

$$\begin{aligned}&h'h'' = -\tfrac{1}{2}h''' \qquad g''' = h''' \qquad h'^2 = -h'' \qquad g' = h' \qquad g'' = h''\\ &h'g'' = -\tfrac{1}{2}g''' \qquad h''g' = -\tfrac{1}{2}g''' \qquad h'g' = -g''.\end{aligned} \tag{8}$$

Having seen the expression for $h(w)$ and $g(w)$ from equations (7) and (8), we investigate $w(x, y, t)$. Substituting equations (7) and (8) into (4) and (5) and setting the coefficients of h''', h'', h', g''', g'' and g' to zero, yield a over-determined system of partial differential equations w.r.t. w:

$$\begin{aligned}&w_x w_y (w_{xx} + 2Aw_x + w_t) = 0\\ &w_y (w_{xx} + 2Aw_x + w_t) + w_x \partial_y (w_{xx} + 2Aw_x + w_t) + w_{xx} (w_{xx} + 2Aw_x + w_t) = 0\\ &\partial_{xy} (w_{xx} + 2Aw_x + w_t) = 0\\ &w_y (w_{xx} + 2Aw_x + w_t) + w_x \partial_y (w_{xx} + 2Aw_x + w_t) + w_{xx} (w_{xx} + 2Aw_x + w_t) + 2Bw_x^2 = 0\\ &\partial_{xy} (w_{xx} + 2Aw_x + w_t) + 2Bw_{xx} = 0\end{aligned} \tag{9}$$

which hold, provided that

$$w_{xx} + 2Aw_x + w_t = 0 \tag{10a}$$

$$B = 0. \tag{10b}$$

One can see that the following simple formal solution for equation (10a),

$$w(x, y, t) = 1 + \exp(\alpha x + \beta y + \gamma t + \delta)$$

would lead to nothing but solitary waves, where α, β, γ and δ are proper constants.

Nevertheless, this clue inspires one to proceed further. More sophisticated than solitary waves is the following x-linear form:

$$w(x, y, t) = P(y, t) + \exp[\Theta(y, t)x + \Psi(y, t)] \tag{11}$$

where $P(y, t)$, $\Theta(y, t)$ and $\Psi(y, t)$ are differentiable functions of y and t only to be determined later. After the substitution of equations (3), (7), (8), (10) and (11) with symbolic computation, we find that two equations of system (1) give rise to the same large equation as follows:

$$\begin{aligned}
&-x^2 e^{\Theta x+\Psi} P\Theta\Theta_y\Theta_t + (e^{\Theta x+\Psi})^2\Theta_{yt} + xe^{\Theta x+\Psi}(P_t\Theta\Theta_y - 2AP\Theta^2\Theta_y - P\Theta^3\Theta_y \\
&\quad + P\Theta\Theta_{yt} + P_y\Theta\Theta_t + 2P\Theta_t\Theta_y - P\Theta\Theta_t\Psi_y - P\Theta\Theta_y\Psi_t) \\
&\quad + e^{\Theta x+\Psi}(-P_{ty}\Theta + 2AP_y\Theta^2 + P_y\Theta^3 - P_t\Theta_y + 4AP\Theta\Theta_y + 3P\Theta^2\Theta_y \\
&\quad + 2P\Theta_{ty} - P_y\Theta_t + P_t\Theta\Psi_y - 2AP\Theta^2\Psi_y - P\Theta^3\Psi_y \\
&\quad + P\Theta_t\Psi_y + P\Theta\Psi_{yt} + P_y\Theta\Psi_t + P\Theta_y\Psi_t - P\Theta\Psi_t\Psi_y) \\
&\quad + P^2\Theta\Theta_t\Theta_y x^2 - P(P_t\Theta\Theta_y - 2AP\Theta^2\Theta_y - P\Theta^3\Theta_y \\
&\quad + P\Theta\Theta_{yt} + P_y\Theta\Theta_t + 2P\Theta_t\Theta_y - P\Theta\Theta_t\Psi_y - P\Theta\Theta_y\Psi_t)x \\
&\quad - PP_{ty}\Theta + 2P_tP_y\Theta_t - PP_y\Theta^3 + P^2\Theta_{ty} - PP_t\Theta_y \\
&\quad + 3P^2\Theta^2\Theta_y - PP_y\Theta_t - PP_t\Theta\Psi_y + P^2\Theta^2\Psi_y + P^2\Theta_t\Psi_y \\
&\quad + P^2\Theta\Psi_{ty} - PP_y\Theta\Psi_t + P^2\Theta_y\Psi_t + P^2\Theta\Psi_t\Psi_y \\
&\quad - 2APP_y\Theta^2 + 4AP^2\Theta\Theta_y + 2AP^2\Theta^2\Psi_y = 0
\end{aligned} \tag{12}$$

which will be satisfied if these terms with $x^2e^{\Theta x+\Psi}$, $xe^{\Theta x+\Psi}$, $(e^{\Theta x+\Psi})^2$, $e^{\Theta x+\Psi}$, x^2, x and x^0 are assumed to vanish separately, that is to say, if we set the coefficients of $x^2e^{\Theta x+\Psi}$, $xe^{\Theta x+\Psi}$, $(e^{\Theta x+\Psi})^2$, $e^{\Theta x+\Psi}$, x^2, x and x^0 to zero. Correspondingly, equation (12) reduces to be a set of constraints, after some algebraic calculations and reductions:

$$\begin{aligned}
&\Theta_y\Theta_t = \Theta_{yt} = 0 \\
&P_t\Theta_y - 2AP\Theta\Theta_y - P\Theta^2\Theta_y + P_y\Theta_t - P\Theta_t\Psi_y - P\Theta_y\Psi_t = 0 \\
&-PP_{ty} + P_tP_y + 2P^2\Theta\Theta_y + P^2\Psi_{ty} + 2AP^2\Theta_y = 0 \\
&P_tP_y - PP_y\Theta^2 - PP_t\Psi_y + P^2\Theta^2\Psi_y - PP_y\Psi_t + P^2\Psi_t\Psi_y - 2APP_y\Theta + 2AP^2\Theta\Psi_y = 0.
\end{aligned} \tag{13}$$

Hence we can obtain two new soliton-like solutions of system (1) as follows.

(I) When $P(y, t) > 0$,

$$\begin{aligned}
H(x, y, t) &= \frac{\partial}{\partial x}h[w(x, y, t)] + A = \frac{\Theta(y, t)e^{\Theta(y,t)x+\Psi(y,t)}}{P(y, t) + e^{\Theta(y,t)x+\Psi(y,t)}} + A \\
&= \frac{\Theta(y, t)}{2}\left\{1 + \tanh\left[\frac{\Theta(y, t)x + \Psi(y, t) - \ln P(y, t)}{2}\right]\right\} + A
\end{aligned} \tag{14}$$

$$\begin{aligned}
G(x, y, t) &= \frac{\partial^2}{\partial x\partial y}g[w(x, y, t)] = \frac{\Theta(y, t)[\Theta_y(y, t)x + \Psi_y(y, t) - P_y(y, t)/P(y, t)]}{4} \\
&\quad \times\operatorname{sech}^2\left[\frac{\Theta(y, t)x + \Psi(y, t) - \ln P(y, t)}{2}\right] \\
&\quad + \frac{\Theta_y(y, t)}{2}\left\{1 + \tanh\left[\frac{\Theta(y, t)x + \Psi(y, t) - \ln P(y, t)}{2}\right]\right\}.
\end{aligned} \tag{15}$$

(II) When $P(y, t) < 0$,

$$H(x, y, t) = \frac{\partial}{\partial x}h[w(x, y, t)] + A = \frac{\Theta(y, t)e^{\Theta(y,t)x+\Psi(y,t)}}{P(y, t) + e^{\Theta(y,t)x+\Psi(y,t)}} + A$$

$$= \frac{\Theta(y,t)}{2}\left\{1+\coth\left[\frac{\Theta(y,t)x+\Psi(y,t)-\ln|P(y,t)|}{2}\right]\right\}+A \tag{16}$$

$$\begin{aligned} G(x,y,t) = \frac{\partial^2}{\partial x\partial y}g[w(x,y,t)] &= \frac{\Theta(y,t)[\Theta_y(y,t)x+\Psi_y(y,t)-P_y(y,t)/P(y,t)]}{4} \\ &\times\operatorname{csch}^2\left[\frac{\Theta(y,t)x+\Psi(y,t)-\ln|P(y,t)|}{2}\right] \\ &+\frac{\Theta_y(y,t)}{2}\left\{1+\coth\left[\frac{\Theta(y,t)x+\Psi(y,t)-\ln|P(y,t)|}{2}\right]\right\}. \end{aligned} \tag{17}$$

Remark. We call it a solitary wave solution if a certain solution contains the variable $\frac{1}{2}(\Theta(y,t)x+\Psi(y,t)-\ln|P(y,t)|)$ which is a linear form of x, y and t: otherwise, we call it a soliton-like solution.

The physical interest of the solutions obtained above lies in the fact that they describe certain soliton-like surface waves. The actual form of the amplitude depends on the choices of $\Theta(y,t)$, $\Psi(y,t)$ and $P(y,t)$, while its horizontal velocity depends on $\Theta(y,t)$. As examples, we shall consider the following cases. (Remark: in the following part, we only consider solutions (14) and (15) further, because solutions (16) and (17) are similar to solutions (14) and (15).)

Case 1. When $\Theta(y,t)=\theta=\text{const}\neq 0$, it is easy to deduce from equation (13) that

$$\begin{aligned} &-PP_{ty}+P_tP_y+P^2\Psi_{ty}=0 \\ &P_tP_y-PP_y\theta^2-PP_t\Psi_y+P^2\Psi_t\Psi_y+P^2\theta^2\Psi_y-PP_y\Psi_t \\ &-2APP_y\theta+2AP^2\theta\Psi_y=0 \end{aligned} \tag{18}$$

and the exact soliton-like solutions can be written as

$$H(x,y,t)=\frac{\theta}{2}\tanh\left[\frac{\theta x+\Psi(y,t)-\ln P(y,t)}{2}\right]+\frac{\theta}{2}+A \tag{19}$$

$$G(x,y,t)=\frac{\theta[\Psi_y(y,t)-P_y(y,t)/P(y,t)]}{4}\operatorname{sech}^2\left[\frac{\theta x+\Psi(y,t)-\ln P(y,t)}{2}\right] \tag{20}$$

where $P(y,t)$ and $\Psi(y,t)$ satisfy equation (18). In this case, there also exist several types of solutions to be considered further:

Case 1a. Soliton-like solution. When $P(y,t)=c_1y+c_2$, $A=0$, we obtain from equation (26) that $\Psi(y,t)=\psi(y)-\theta^2t$. Thus the corresponding solutions of system (1) are

$$H(x,y,t)=\frac{\theta}{2}\left\{1+\tanh\left[\frac{\theta x+\psi(y)-\theta^2t-\ln(c_1y+c_2)}{2}\right]\right\} \tag{21}$$

$$G(x,y,t)=\frac{\theta[\psi_y(y)-c_1/(c_1y+c_2)]}{4}\operatorname{sech}^2\left[\frac{\theta x+\psi(y)-\theta^2t-\ln(c_1y+c_2)}{2}\right]. \tag{22}$$

Case 1b. Solutions independent of y. When $P(y,t)=c_1y+c_2$, $A\neq 0$ with arbitrary constants c_1 and c_2, we obtain from equation (20) that $\Psi(y,t)=\ln(c_1y+c_2)-\theta^2t$. Thus the corresponding solutions independent of y for system (1) are

$$H(x,y,t)=\frac{\theta}{2}\left\{1+\tanh\left[\frac{\theta x-\theta^2t}{2}\right]\right\}+A \tag{23}$$

$$G(x,y,t)=0. \tag{24}$$

Case 1c. Solitary waves. Let us take that

$$\Psi(y,t) = \alpha y + \beta t + \gamma \qquad P(y,t) = 1 \tag{25}$$

with α, β and γ constants. Substituting equation (25) into (18) we get

$$A = -\frac{\beta + \theta^2}{2\theta}.$$

Therefore we have solitary wave solutions of system (1):

$$H(x,y,t) = \frac{\theta}{2}\left\{1 + \tanh\left[\frac{\theta x + \alpha y + \beta y + \gamma}{2}\right]\right\} - \frac{\beta + \theta^2}{2\theta} \tag{26}$$

$$G(x,y,t) = \frac{\alpha\theta}{4}\operatorname{sech}^2\left[\frac{\theta x + \alpha y + \beta y + \gamma}{2}\right] \tag{27}$$

in which the variable $\frac{1}{2}(\theta x + \alpha y + \beta y + \gamma)$ is a linear form of x, y and t. Thus we call them solitary wave solutions of system (1). It is clear that they are nothing other than a special case of solutions (21) and (22).

Case 1d. Soliton-like solution. Taking $P(y,t) = c_3 \mathrm{e}^t + c_4$ with arbitrary constants c_3 and c_4, the first equation of system (24) becomes

$$\Psi(y,t) = 0 \qquad \text{i.e.} \quad \Psi(y,t) = \alpha(y) + \beta(t) \tag{28}$$

where $\alpha(y)$ and $\beta(t)$ are arbitrary, differentiable functions. Substituting equation (28) into the second equation of system (18) yields

$$\Psi(y,t) = \alpha(y) - (\Theta^2 + 2A\Theta - 1)t. \tag{29}$$

As a result, we get another solution of system (1):

$$H(x,y,t) = \frac{\theta}{2}\left\{1 + \tanh\left[\frac{\theta x + \alpha(y) - (\theta^2 + 2A\theta - 1)t - \ln(c_3\mathrm{e}^t + c_4)}{2}\right]\right\} + A \tag{30}$$

$$G(x,y,t) = \frac{\theta\alpha_y(y)}{4}\operatorname{sech}^2\left[\frac{\theta x + \alpha(y) - (\theta^2 + 2A\theta - 1)t - \ln(c_3\mathrm{e}^t + c_4)}{2}\right] \tag{31}$$

in which the variable $\frac{1}{2}[\theta x + \alpha(y) - (\theta^2 + 2A\theta - 1)t - \ln(c_3\mathrm{e}^t + c_4)]$ is a nonlinear form of y and t. Thus we call them soliton-like solutions of system (1).

Case 2. When $\Theta_y(y,t) \neq 0$, $\Theta_t(y,t) = 0$, i.e. $\Theta(y,t) = \theta(y)$, it is easy to deduce from equation (19) that

$$\begin{aligned} &P_t - 2AP\theta - P\theta^2 - P\Psi_t = 0 \\ &-PP_{ty} + P_tP_y + 2P^2\theta\theta_y + P^2\Psi_{ty} + 2AP^2\theta_y = 0 \end{aligned} \tag{32}$$

and we have

$$H(x,y,t) = \frac{\theta(y)}{2}\left\{1 + \tanh\left[\frac{\Theta(y)x + \Psi(y,t) - \ln P(y,t)}{2}\right]\right\} + A \tag{33}$$

$$\begin{aligned} G(x,y,t) = {} & \frac{\theta(y)[\theta_y(y)x + \Psi_y(y,t) - P_y(y,t)/P(y,t)]}{4} \\ & \times \operatorname{sech}^2\left[\frac{\theta(y)x + \Psi(y,t) - \ln P(y,t)}{2}\right] \\ & + \frac{\theta_y(y)}{2}\left\{1 + \tanh\left[\frac{\theta(y)x + \Psi(y,t) - \ln P(y,t)}{2}\right]\right\}. \end{aligned} \tag{34}$$

Case 2a. Taking $P(y, t) = p_1(y)p_2(t)$ with the arbitrary functions $p_1(y)$ and $p_2(t)$, equation (32) can rewritten as

$$\begin{aligned} &\frac{p_{2t}(t)}{p_2(t)} - 2A\theta - \theta^2 - \Psi_t = 0 \\ &2\theta\theta_y + \Psi_{ty} + 2A\theta_y = 0. \end{aligned} \tag{35}$$

It is noted that the integration of the second equation of system (35) over y leads to the first equation of system (35). The second time of integration over t further leads to the expression

$$\Psi(y, t) = -\theta^2(y)t - 2A\theta(y)t + \psi(y) + \ln p_2(t) \tag{36}$$

where $\psi(y)$ is an arbitrary function of y. Therefore we have

$$H(x, y, t) = \frac{\theta(y)}{2}\left\{1 + \tanh\left[\frac{\theta(y)x - \theta^2(y)t - 2A\theta(y)t + \psi(y) - \ln p_1(y)}{2}\right]\right\} + A \tag{37}$$

$$\begin{aligned} G(x, y, t) = &\frac{\theta(y)[\theta_y(y)x - 2\theta(y)\theta_y(y)t - 2A\theta_y(y)t + \psi_y(y) - p_{1y}(y)/p_1(y)]}{4} \\ &\times \operatorname{sech}^2\left[\frac{\theta(y)x - \theta^2(y)t - 2A\theta(y)t + \psi(y) - \ln p_1(y)}{2}\right] \\ &+ \frac{\theta_y(y)}{2}\left\{1 + \tanh\left[\frac{\theta(y)x - \theta^2(y)t - 2A\theta(y)t + \Psi(y) - \ln p_1(y)}{2}\right]\right\} \end{aligned} \tag{38}$$

where $\theta(y)$, $p_1(y)$ and $\psi(y)$ are all arbitrary, differentiable functions of y.

Case 2b. Taking $P(y, t) = p_1(y) + p_2(t)$ with the arbitrary functions $p_1(y)$ and $p_2(t)$, equation (32) can rewritten as

$$\begin{aligned} &\frac{p_{2t}(t)}{p_1(y) + p_2(t)} - 2A\theta - \theta^2 - \Psi_t = 0 \\ &p_{1y}p_{2t} + [p_1(y) + p_2(t)] + 2\theta\theta_y + \Psi_{ty} + 2A\theta_y = 0. \end{aligned} \tag{39}$$

It is noted that the integration of the second equation of system (35) over y leads to the first equation of system (41). The second time of integration, over t, further leads to the expression

$$\Psi(y, t) = -\theta^2(y)t - 2A\theta(y)t + \psi(y) + \ln(p_1(y) + p_2(t)) \tag{40}$$

where $\psi(y)$ is an arbitrary function of y. Thus we get

$$H(x, y, t) = \theta(y)\left\{1 + \tanh\left[\frac{\theta(y)x + -\theta^2(y)t - 2A\theta(y)t + \psi(y)}{2}\right]\right\} + A \tag{41}$$

$$\begin{aligned} G(x, y, t) = &\frac{\theta(y)[\theta_y(y)x - 2\theta(y)\theta_y(y)t - 2A\theta_y(y)t + \psi_y(y)]}{4} \\ &\times \operatorname{sech}^2\left[\frac{\theta(y)x - \theta^2(y)t - 2A\theta(y)t + \psi(y)}{2}\right] \\ &+ \frac{\theta_y(y)}{2}\left\{1 + \tanh\left[\frac{\theta(y)x - \theta^2(y)t - 2A\theta(y)t + \psi(y)}{2}\right]\right\} \end{aligned} \tag{42}$$

where $\theta(y)$ and $\psi(y)$ are all arbitrary, differentiable functions of y. Though $P(y, t)$ are different between case 2a and case 2b, the solutions (41) and (42) are the special cases of the solutions (37) and (38).

Case 3. When $\Theta_t(y,t) \neq 0$, $\Theta_y(y,t) = 0$, i.e. $\Theta(y,t) = \theta(t)$, it is easy to deduce from equation (13) that

$$\begin{aligned} &P_y - P\Psi_y = 0 \\ &-PP_{ty} + P_t P_y + P^2\Psi_{ty} = 0 \end{aligned} \tag{43}$$

which leads to

$$\Psi(y,t) = \psi_1(y) + \psi_2(t) \qquad P(y,t) = P(t)\exp[\psi_1(y) + \psi_2(t)] \tag{44}$$

where $\psi_1(y)$, $\psi_2(t)$ and $P(t)$ are arbitrary functions. Therefore we have

$$H(x,y,t) = \frac{\theta(t)}{2}\left\{1 + \tanh\left[\frac{\theta(t)x - \ln P(t)}{2}\right]\right\} + A \tag{45}$$

$$G(x,y,t) = 0. \tag{46}$$

In this case, the solution $H(x,y,t)$ is independent of y, and $G(x,y,t)$ is zero.

4. Summary and conclusions

In summary, we have obtained many families of soliton-like solutions for the BK equation in $(2+1)$-dimensional spaces by using the symbolic computation method developed by Gao and Tian. This method plays an important role in studying many properties of NEEs in soliton theory. These solutions obtained may be of important significance for the explanation of some practical physical problems.

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References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Miura M R 1978 *Backlund Transformation* (Berlin: Springer)
[3] Gu C H *et al* 1995 *Soliton Theory and its Applications* (Berlin: Springer)
[4] Lou S Y 1997 *NBN-IMP* **27** 29
[5] Liu Q P 1995 *Phys. Lett.* A **198** 178
[6] Weiss J, Tabor M and Carnevale J 1983 *J. Math. Phys.* **24** 552
[7] Lou S Y and Hu X B 1994 *J. Phys. A: Math. Gen.* **27** 207
[8] Ruan H Y and Chen Y X 1998 *Acta Phys. Sin. (Overseas Edition)* **7** 241
[9] Yan Z Y and Zhang H Q 1999 *Commun. Nonlinear Sci. Numer. Simul.* **4** 146
[10] Sirendaoreji 1999 *J. Phys. A: Math. Gen.* **32** 6897
[11] Gao Y and Tian B 1997 *Comput. Math. Appl.* **33** 115
[12] Tian B and Gao Y 1996 *J. Phys. A: Math. Gen.* **29** 2895
[13] Gao Y and Tian B 1995 *Comput. Math. Appl.* **30** 97
[14] Yan Z Y and Zhang H Q 2000 *Appl. Math. Mech.* **20** 385
[15] Hong W and Oh K 2000 *Comput. Math. Appl.* **39** 29
[16] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[17] Hirota R 1979 *J. Phys. Soc. Japan* **46** 312
[18] Nucci M and Clarkson P 1992 *Phys. Lett.* A **164** 49